Eur. Phys. J. D **25**, 123–128 (2003)
DOI: 10.1140/epjd/e2003-00237-y

The European Physical Journal D

Quantization for brachistochrone problem

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Received 18 July 2002 / Received in final form 29 January 2003
Published online 30 July 2003 –

Abstract. The brachistochrone curve corresponds to the minimization of the time functional. In this paper we discuss the dynamics of a massive particle, which moves classically on the brachistochrone curve under the potential $V = -mgy$. We derive the Lagrangian and the Hamiltonian of the particle and show that this problem corresponds to the particle in an infinite wall with a harmonic oscillator potential and the solutions of Schrödinger's equation are confluent hypergeometric functions. We also discuss the periodic potential problem for the brachistochrones and obtain the band structure of Kronig-Penney model for the particle with positive energy in a certain limit.

PACS. 03.65.-w Quantum mechanics – 03.65.Ge Solutions of wave equations: bound states – 71.15.Ap Basis sets (LCAO, plane-wave, APW, etc.) and related methodology (scattering methods, ASA, linearized methods, etc.)

1 Introduction

Brachistochrone is one of the oldest problems of physics and also one of the earliest problems proposed in the calculus of variations; the story of its dates back to 300-years [1,2], and it is known as "the shortest time problem of a particle moving between two points on a vertical plane". First, Bernoulli proposed the problem and solved it by using the optical analogy of Fermat's least-time principle and the method of the calculus of variations. Many famous mathematicians of the time paid attention to this problem and the attempts on the subject are considered as the fundamentals of the calculus of variations. The well-known solution for the classical least time or brachistochrone problem is a cycloid, which is a curve described by a point P on a circle that rolls without slipping. The period of oscillation for the brachistochrone curve depends on the amplitude.

Even now, it is very popular, because it has several generalizations, including, *e.g.* classical Newtonian generalization, relativistic generalizations [3,8]. It has been generalized to general central forces, both attractive and repulsive [9]. There has been some new approaches to obtain analytic solutions of the brachistochrone problem [10,11].

The aim of this study is to discuss the quantum dynamics of a particle, which moves classically on the brachistochrone curve in a homogeneous force field. The brachistochrone curve corresponds to the minimization of the time functional. In Section 2, we are interested in to derive the Lagrangian and the Hamiltonian of the particle, which moves also on the brachistochrone curve by the minimization of the action functional. In Section 3, we discuss the solution of the Schrödinger equation for this Hamiltonian and derive the energy eigenvalues and eigenfunctions. In Section 4, we consider the periodic extension of the brachistochrone curves and discuss the solution of the Schrödinger equation for the particle in the periodic extension of the original brachistochrone problem.

2 Classical system

We consider a particle of mass m moving on a brachistochrone curve, so that the net force on it is the constant force of gravity, $-mg\widehat{y}$. The minimization of the time function, t gives the following parametric solutions for the x and y-coordinates of the particle:

$$\begin{aligned} x &= a\,(\theta - \sin\theta) \\ y &= a\,(1 - \cos\theta)\,, \end{aligned} \tag{1}$$

where, the variable θ and the constant a are the angle of rotation and the radius of the circle rolling on the x-axis, respectively. We write the Lagrangian of the particle which moves classically on the curve defined by the parametric equations (1) with the potential energy $V = -mgy$ and the kinetic energy $E_k = m(\dot{x}^2 + \dot{y}^2)/2$. It is

$$L\left(\theta, \dot{\theta}\right) = (1 - \cos\theta)\left(ma^2\dot{\theta}^2 + mga\right)\cdot$$

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This is the Lagrangian of the particle on the cycloid. We change the variable θ:

$$u = a\cos\frac{\theta}{2}\cdot$$

Then, the Lagrangian L becomes

$$L\left(u,\dot{u}\right) = 8m\left[\dot{u}^2 + \omega^2\left(a^2 - u^2\right)\right], \tag{2}$$

where $\omega^2 = g/4a$. Equation (2) is the Lagrangian of the harmonic oscillator with the effective mass $\mu = 16m$, with angular frequency ω. Because of the transformation from θ to u, the variable u is restricted by the condition $|u| \leq a$. In some sense the brachistochrone problem corresponds to the harmonic oscillator problem with the infinite wells at $u = \pm a$. The classical equation of motion is

$$\ddot{u} + \omega^2 u = 0. \tag{3}$$

So, classically we obtain the following solutions:

$$u\left(t\right) = A\sin\omega t + B\cos\omega t, \tag{4}$$

and

$$\theta\left(t\right) = 2\cos^{-1}\left(\frac{A\sin\omega t + B\cos\omega t}{a}\right). \tag{5}$$

The conjugate momentum of the system is

$$p_u = \frac{\partial L}{\partial \dot{u}} = 16m\dot{u}.$$

Thus, the Hamiltonian of system can be written as

$$H = \frac{p_u^2}{2\mu} - \frac{\mu\omega^2}{2}\left(a^2 - u^2\right), \tag{6}$$

where $\mu = 16m$ and $-a \leq u \leq a$, because of the geometry of the problem. That means there are infinite well potential for $|u| \geq a$.

3 Quantization

The energy eigenfunctions, Ψ_E satisfy the following eigenvalue equation:

$$\widehat{H}\Psi_E(\mathbf{r}) = E\Psi_E(\mathbf{r}), \tag{7}$$

where E is the energy eigenvalue and $\widehat{H}$ is the Hamiltonian operator corresponding to equation (6). We write equation (7) explicitly

$$\frac{\mathrm{d}^2\psi\left(u\right)}{\mathrm{d}u^2} + \frac{2\mu}{\hbar^2}\left[E + \frac{\mu\omega^2}{2}\left(a^2 - u^2\right)\right]\psi\left(u\right) = 0. \tag{8}$$

Here, we assume that

$$E > -\frac{1}{2}\mu\omega^2 a^2,$$

where, $-\mu\omega^2 a^2/2$ is the minimum value of the potential energy. For the sake of simplicity we introduce the following dimensionless parameters in equation (8):

$$\lambda = \frac{E'}{\hbar\omega}, \quad E' = E + \frac{1}{2}\mu\omega^2 a^2, \quad \varsigma = \sqrt{\frac{2\mu\omega}{\hbar}}u. \tag{9}$$

Thus, equation (8) reduces to

$$\frac{\mathrm{d}^2\psi\left(\varsigma\right)}{\mathrm{d}\varsigma^2} + \left(\lambda - \frac{\varsigma^2}{4}\right)\psi\left(\varsigma\right) = 0. \tag{10}$$

Equation (10) is known as the oscillator differential equation and the solution of it are well known in the infinite interval $(-\infty,\infty)$:

$$\psi\left(\varsigma\right) = \mathrm{e}^{-\frac{\varsigma^2}{4}}\left[C\,{}_1F_1\left(-\frac{p}{2},\frac{1}{2};\frac{\varsigma^2}{2}\right)\right.$$
$$\left. + C'\varsigma\,{}_1F_1\left(\frac{1-p}{2},\frac{3}{2};\frac{\varsigma^2}{2}\right)\right], \tag{11}$$

where $p = \lambda - 1/2$. C and C' are the constants.

Since the potential energy is an even function of ς, $\psi\left(-\varsigma\right)$ is also an eigenfunction corresponding to the same energy eigenvalue like $\psi\left(\varsigma\right)$. Therefore, the general solutions have definite parities:

$$\Psi^{(+)}\left(\varsigma\right) = C\mathrm{e}^{-\frac{\varsigma^2}{4}}\,{}_1F_1\left(-\frac{p}{2},\frac{1}{2};\frac{\varsigma^2}{2}\right)$$
$$\Psi^{(-)}\left(\varsigma\right) = C'\mathrm{e}^{-\frac{\varsigma^2}{4}}\varsigma\,{}_1F_1\left(\frac{1-p}{2},\frac{3}{2};\frac{\varsigma^2}{2}\right). \tag{12}$$

The geometry of the problem gives the boundary conditions as

$$\Psi^{(\pm)}\left(\pm\sqrt{\frac{2\mu\omega}{\hbar}}a\right) = 0. \tag{13}$$

This condition gives

$${}_1F_1\left(-\frac{p}{2},\frac{1}{2};\frac{\mu\omega a^2}{\hbar}\right) = {}_1F_1\left(\frac{1-p}{2},\frac{3}{2};\frac{\mu\omega a^2}{\hbar}\right) = 0. \tag{14}$$

It is difficult to calculate the roots of equation (14); because each of them represents infinite series. For this reason we use a graphical method. Firstly, we write their first twenty five terms of the series for each of the confluent hypergeometric functions in equation (14). Later, we draw the graphs of these functions for the each different numerical value given for $\mu\omega a^2/\hbar$. So we obtain numerically p points at the zeros of the function for each $\mu\omega a^2/\hbar$, and then the tables of data $\left(\mu\omega a^2/\hbar, p\right)$. It is shown in Figure 1. To fit an equation to these data, we use a process called curve fitting [12]; the resulting equation is given below:

$$p_n = n + \frac{\left(n+1\right)^2\hbar}{\mu\omega a^2}; \qquad n = 0,1,2,...\,(\text{integers}). \tag{15}$$

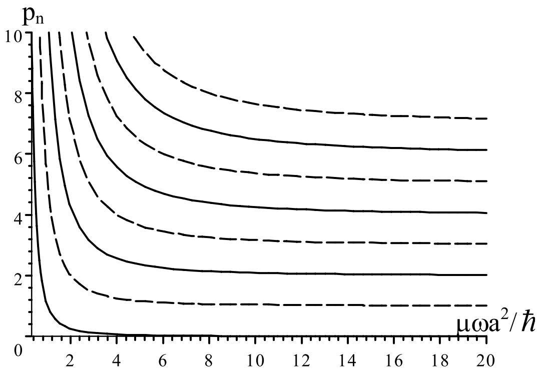

Fig. 1. The first 8 p_n eigenvalues mentioned in the brachistochrone problem in the potential, $V = -mgy$ as a function of the parameter $\mu\omega a^2/\hbar$. The solid curves belong to p_n eigenvalues with even subindexes; and the dashed curves belong to p_n eigenvalues with odd subindexes.

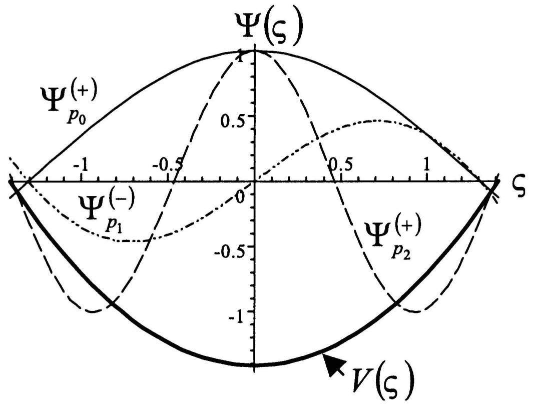

Fig. 2. The first 3 exact wave functions, but taking the approximate energy eigenvalues, p_n for the brachistochrone problem. The dark curve shows the schematic of the potential as a function of ς. We set here $\mu\omega a^2/\hbar = 1$ and, $n = 0$ for the light solid curve, $n = 1$ for the dot dashed curve, $n = 2$ for the dashed curve.

Thus, we can write unnormalized eigenfunctions and the energy eigenvalues as

$$\Psi^{(+)}_{p_n}(\varsigma) = C\mathrm{e}^{-\frac{\varsigma^2}{4}}\,{}_1F_1\left(-\frac{p_n}{2},\frac{1}{2};\frac{\varsigma^2}{2}\right)$$

$$E^{(+)}_{p_n} = -\frac{1}{2}\mu\omega^2a^2 + \hbar\omega\left(p_n+\frac{1}{2}\right);\ n = \text{even integers}, \tag{16}$$

$$\Psi^{(-)}_{p_n}(\varsigma) = C'\mathrm{e}^{-\frac{\varsigma^2}{4}}\varsigma\,{}_1F_1\left(\frac{1-p_n}{2},\frac{3}{2};\frac{\varsigma^2}{2}\right)$$

$$E^{(-)}_{p_n} = -\frac{1}{2}\mu\omega^2a^2 + \hbar\omega\left(p_n+\frac{1}{2}\right);\ n = \text{odd integers}. \tag{17}$$

Taking approximate energy eigenvalues, the exact forms of the wave functions in equations (16, 17) are plotted in Figure 2.

It is useful to discuss the limits of these solutions for $\omega \to \infty$ and $\omega \to 0$.

(i) The limit $\sqrt{2\mu\omega/\hbar}\,a \to \infty$ corresponds to the harmonic oscillator problem. As it is given in equation (15) and shown in Figure 1, if $\sqrt{2\mu\omega/\hbar}\,a$ is very large, these solutions give the integer values for p. If we look at the asymptotic behaviour of $\Psi_p^{(\pm)}$ as $\sqrt{2\mu\omega/\hbar}\,a \to \pm\infty$ [13], we can check these integer values for p. Using these integer values for p and the following connection between confluent hypergeometric function and Hermite polynomials [14]:

$$H_{2n}(x) = \frac{(-1)^n\,(2n)!}{n!}\,{}_1F_1\left(-n,\frac{1}{2};x^2\right) \tag{18}$$

$$H_{2n+1}(x) = \frac{(-1)^n\,2\,(2n+1)!}{n!}x\,{}_1F_1\left(-n,\frac{3}{2};x^2\right), \tag{19}$$

so we obtain the following normalized eigenfunctions and the energy eigenvalues as

$$\varphi_n(u) = \left(\frac{\sqrt{\pi}}{2^n n!}\right)^{\frac{1}{2}} H_n\left(\sqrt{\frac{\mu\omega}{\hbar}}u\right)\mathrm{e}^{-\frac{\mu\omega}{2\hbar}u^2}$$

$$E'_n = \hbar\omega\left(n+\frac{1}{2}\right);\qquad n = p_n = 0,1,2,..., \tag{20}$$

where $H_n\left(\varsigma/\sqrt{2}\right)$ are the Hermite polynomials. As it is seen from equation (20), the eigenfunctions and the energy levels obtaining in this limit are the same as those of a harmonic oscillator solutions.

(ii) The condition $\sqrt{2\mu\omega/\hbar}\,a \ll 1$ represents the case of an infinite well potential. To show this we apply the power series expansion of $\Psi^{(\pm)}(\varsigma)$ around $\varsigma = \pm\sqrt{2\mu\omega/\hbar}\,a$ in equation (11)

$$\Psi^{(\pm)}(\varsigma) = N^{(\pm)}\sum_{n=0}^{\infty}\left.\frac{\mathrm{d}^n\Psi^{(\pm)}(\varsigma)}{\mathrm{d}\varsigma^n}\right|_{\varsigma=\sqrt{\frac{2\mu\omega}{\hbar}}a} \times\frac{\left(\varsigma-\sqrt{\frac{2\mu\omega}{\hbar}}a\right)^n}{n!}. \tag{21}$$

For $\omega \to 0$, the series in equation (21) reduce to

$$\lim_{\omega\to 0}\Psi^{(\pm)}(\varsigma) = \Omega^{(\pm)}(\varsigma) \cong \frac{N^{(\pm)}}{\sqrt{\lambda}}\begin{cases}\cos(\sqrt{\lambda}\varsigma) & \text{for even solutions}\\ \sin(\sqrt{\lambda}\varsigma) & \text{for odd solutions.}\end{cases} \tag{22}$$

If we use the continuity conditions for the wave functions at the boundaries of the classical region $\pm\sqrt{2\mu\omega/\hbar}\,a$, and normalize them in interval $[-\sqrt{2\mu\omega/\hbar}\,a, \sqrt{2\mu\omega/\hbar}\,a]$, we obtain the normalized eigenfunctions and eigenvalues of energy which are the solutions of the infinite-well potential as

$$\Omega^{(\pm)}(u) \cong \frac{1}{\sqrt{a}}\begin{cases}\cos\left(\frac{(n+1/2)\pi}{a}u\right),\ E_n'^{(+)} = \frac{(2n+1)^2\hbar^2\pi^2}{8\mu a^2}\\ \text{for even solutions.}\\ \sin\left(\frac{n\pi}{a}u\right),\ E_n'^{(-)} = \frac{n^2\hbar^2\pi^2}{2\mu a^2}\\ \text{for odd solutions.}\end{cases} \tag{23}$$

where, $n = 0, 1, 2, ...$

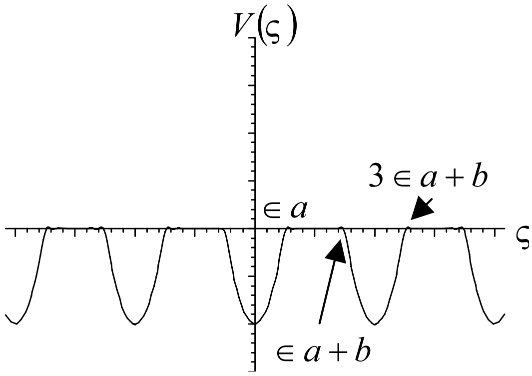

Fig. 3. Schematic of the periodic potential for brachistochrones in one dimension.

4 Periodic potential problem for brachistochrones

For a potential with period, P the solutions of the Schrödinger's equation satisfy the Bloch condition and the spectrum shows energy bands, and the wave functions satisfy the periodicity condition, $\Psi(\varsigma) = \pm\Psi(\varsigma + P)$ at the band edges. The potential of our problem is in the form of $V = -\mu\omega^2(a^2 - \varsigma^2/\epsilon^2)/2$, where $\epsilon^2 = 2\mu\omega/\hbar$. We consider an infinite array of these cycloid shaped potentials with the interval b in one dimension as it is shown in Figure 3. The particle moves under the influence of these periodic potentials. Thus the potential is the symmetric and the form of the following periodic function:

$$V(\varsigma + nP) = V(\varsigma), \tag{24}$$

where $P = 2\epsilon a + b$ and b is the length of the distance in which the potential is zero. Thus the solutions of the Schrödinger equation are also periodic according to Floquet theory [13] and given as

$$\Psi(\varsigma + nP) = \mathrm{e}^{in\Phi}\Psi(\varsigma), \tag{25}$$

where $\mathrm{e}^{\mathrm{i}\Phi}$ is a phase factor. We will first discuss the energy eigenvalues for bound and unbound states, then present some limit cases for $b \to 0$; and $2\epsilon a \to 0$, $V_0 = -\mu\omega^2 a^2/2 \to -\infty$.

4.1 Bound states

The energies of the bound states are in the interval $-(\mu\omega^2/2)a^2 < E < 0$, and the eigenfunctions have the following form:

In the period $-\epsilon a < \varsigma < b + \epsilon a$,

$$\Psi(\varsigma) = \begin{cases} M\Psi^{(+)}(\varsigma) + N\Psi^{(-)}(\varsigma);\ -\epsilon a < \varsigma < \epsilon a \\ A\mathrm{e}^{\kappa\varsigma} + B\mathrm{e}^{-\kappa\varsigma};\ \epsilon a < \varsigma < b + \epsilon a, \end{cases} \tag{26}$$

where $\Psi^{(+)}(\varsigma)$ and $\Psi^{(-)}(\varsigma)$ are defined in equation (12), and M, N, A and B are the constants. The second solution in equation (26) is the solutions of Schrödinger's equation with the zero potential, and $\kappa^2 = |E|/\hbar\omega$.

In the next period, $b + \epsilon a < \varsigma < 2b + 3\epsilon a$ or $b + \epsilon a < \varsigma < b + \epsilon a + P$,

$$\Psi(\varsigma) = \mathrm{e}^{\mathrm{i}\Phi}\begin{cases} M\Psi^{(+)}(\varsigma - P) + N\Psi^{(-)}(\varsigma - P); \\ b + \epsilon a < \varsigma < b + 3\epsilon a \\ A\mathrm{e}^{\kappa(\varsigma - P)} + B\mathrm{e}^{-\kappa(\varsigma - P)}; \\ b + 3\epsilon a < \varsigma < 2b + 3\epsilon a. \end{cases} \tag{27}$$

Thus we use the continuity conditions of the functions, Ψ and of the derivations of them; since the potentials are finite. From the continuity conditions at $\varsigma = \epsilon a$ and $\varsigma = b + \epsilon a$, we obtained the following matrix equations:

$$\begin{pmatrix} c & \epsilon a f & 1 & -1 \\ -\epsilon a\left(\frac{c}{2} + pd\right) & -\frac{\epsilon^2 a^2 f}{2} + f + \frac{\epsilon^2 a^2 (1-p) h}{3} & \kappa & \kappa \\ \mathrm{e}^{\mathrm{i}\Phi} c & -\mathrm{e}^{\mathrm{i}\Phi}\epsilon a f & \mathrm{e}^{\kappa b} & -\mathrm{e}^{-\kappa b} \\ \mathrm{e}^{\mathrm{i}\Phi}\epsilon a\left(\frac{c}{2} + pd\right) & \mathrm{e}^{\mathrm{i}\Phi}\left(-\frac{\epsilon^2 a^2 f}{2} + f + \frac{(1-p)\epsilon^2 a^2 h}{3}\right) & \kappa\mathrm{e}^{\kappa b} & \kappa\mathrm{e}^{-\kappa b} \end{pmatrix} \times \begin{pmatrix} M \\ N \\ A \\ B \end{pmatrix} = 0, \tag{28}$$

where c, f, d and h represent the following constants:

$$c = {}_1F_1\left(-\frac{p}{2}, \frac{1}{2}; \frac{\epsilon^2 a^2}{2}\right)$$
$$f = {}_1F_1\left(\frac{1-p}{2}, \frac{3}{2}; \frac{\epsilon^2 a^2}{2}\right)$$
$$d = {}_1F_1\left(\frac{2-p}{2}, \frac{3}{2}; \frac{\epsilon^2 a^2}{2}\right)$$
$$h = {}_1F_1\left(\frac{3-p}{2}, \frac{5}{2}; \frac{\epsilon^2 a^2}{2}\right)\cdot \tag{29}$$

Equation (28) has untrivial solutions under the following condition:

$$\cos\Phi = \cosh\kappa b - \frac{\epsilon^2 a^2 f(c + 2pd)\cosh\kappa b}{\left[cf - \frac{1}{3}c\epsilon^2 a^2 h(p-1) + \epsilon^2 a^2 pdf\right]} - \frac{\epsilon a b}{2}\left\{1 + \frac{\left[-\left(4\kappa^2 + \epsilon^2 a^2\right)cf + 4\left(1 - \epsilon^2 a^2\right)pdf\right.}{2\left[cf + \epsilon^2 a^2 pdf - \frac{1}{3}c\epsilon^2 a^2 h(p-1)\right]} - \frac{\left.\frac{4}{3}\epsilon^2 a^2 p(p-1)dh\right]}{2\left[cf + \epsilon^2 a^2 pdf - \frac{1}{3}c\epsilon^2 a^2 h(p-1)\right]}\right\}\frac{\sinh\kappa b}{\kappa b}\cdot \tag{30}$$

We can represent the right side of equation (30) as a function of E, $f(E)$, because the energy appears through κ, p, c, f, d, h. Since $|\cos\Phi| \le 1$, the problem has no solution for $|f(E)| > 1$, and hence no allowed energy eigenvalues exist.Thus, we determine the possible energy eigenvalues by the condition

$$-1 \le f(E) \le 1. \tag{31}$$

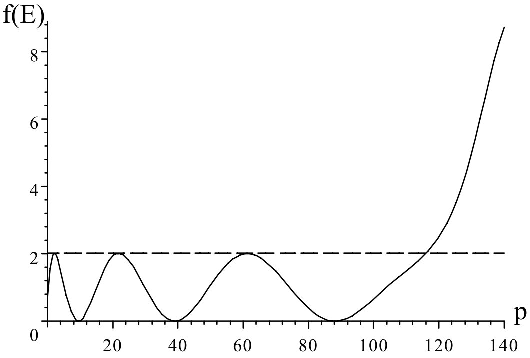

Fig. 4. Allowed $p = -|E|/\hbar\omega - 1/4$ values for $\epsilon a = 1$ at the limit $b \to 0$.

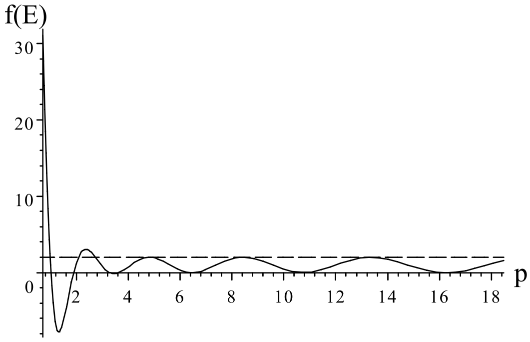

Fig. 5. Allowed $p = -|E|/\hbar\omega + 1/2$ values for $\epsilon a = 2$ at the limit $b \to 0$.

We derive the band structure of the energy spectrum. To understand the structure of the energy spectrum, we can discuss the particular cases for such $b \to 0$ and $2\epsilon a \to 0$, $V_0 = -\frac{1}{2}\mu\omega^2 a^2 \to -\infty$.

4.1.1 Limiting cases

(i) $b \to 0$: in this case, the associated potential for brachistochrones has period $P = 2\epsilon a$. Hence the corresponding energy function is

$$f(E) = \frac{\epsilon^2 a^2 f(c + 2pd)}{\left[cf - \frac{1}{3}c\epsilon^2 a^2 h(p-1) + \epsilon^2 a^2 pdf\right]}. \tag{32}$$

In Figures 4 and 5, we plot the graphics, $(p, f(E))$ for the arbitrary constant values of ϵa. The allowed regions lie in the interval $0 \le f(E) \le 2$. If ϵa is in the interval $(0, 1)$, all regions are allowed and energy spectrum is continuous. For $\epsilon a = 1$, as it is seen from Figure 5, the forbidden energy regions appear for small energies and the spectrum is continuous as the energy increases in the allowed region. For $\epsilon a = 2$, as it is seen from Figure 4, the width of the allowed region decreases and the energy spectrum is continuous in the allowed region. The width of the allowed region decreases, if we use the larger constant values of ϵa. There is no solution for $\epsilon a \gg 1$.

(ii) $2\epsilon a \to 0$, $V_0 = -\mu\omega^2 a^2/2 \to -\infty$: in this case, we reduce the thickness of the potential to zero but we increase the deepness of the potential to infinite, so that the area under the potential function remains constant:

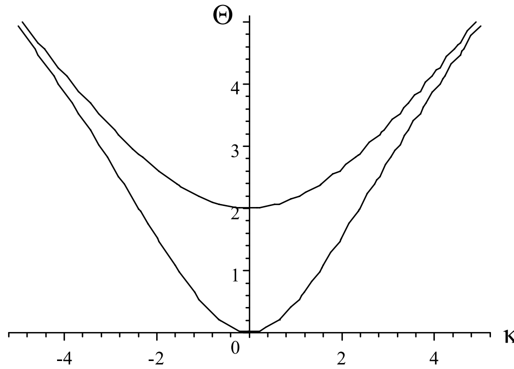

Fig. 6. Allowed regions lie between two curves at the limit $2\epsilon a \to 0$ and $V_0 \to -\infty$.

$$\lim_{\substack{2\epsilon a \to 0 \\ -1/2\mu\omega^2 a^2 \to -\infty}} \frac{2}{3}\mu\omega^2 \epsilon a^3 = \Theta = \text{constant}. \tag{33}$$

Thus, we obtain the following condition for allowed energy values:

$$-1 \le \cosh(\kappa b) - \Theta \frac{\sinh(\kappa b)}{\kappa b} \le 1. \tag{34}$$

As it is seen from Figure 6, if $0 < \Theta \le 2$, the energy eigenvalue spectrum lies between definite values and it is continuous in this interval. If $\Theta > 2$, the energy eigenvalue spectrum contains two allowed bands; as the values of Θ increase, the width of these bands decreases.

4.2 Positive energies

We derive the bound states with positive energy by substituting $k \to -\mathrm{i}\kappa$ in equations (26–28). Then the condition (30) becomes

$$\begin{aligned}\cos\Phi = \cos kb &- \frac{\epsilon^2 a^2 f(c + 2pd)\cos kb}{\left[cf - \frac{1}{3}c\epsilon^2 a^2 h(p-1) + \epsilon^2 a^2 pdf\right]} \\ &- \frac{\epsilon ab}{2}\left\{1 + \frac{\left[\left(4k^2 - \epsilon^2 a^2\right)cf + 4\left(1 - \epsilon^2 a^2\right)pdf\right.}{2\left[cf + \epsilon^2 a^2 pdf - \frac{1}{3}c\epsilon^2 a^2 h(p-1)\right]}\right. \\ &\left. - \frac{\left.\frac{4}{3}\epsilon^2 a^2 p(p-1)dh\right]}{2\left[cf + \epsilon^2 a^2 pdf - \frac{1}{3}c\epsilon^2 a^2 h(p-1)\right]}\right\}\frac{\sin kb}{kb},\end{aligned} \tag{35}$$

where $k^2 = E/\hbar\omega \ge 0$.

4.2.1 Limiting cases

(i) $b \to 0$: the allowed regions are the same with bound states. They differ from each others with the sign of energies. The allowed regions lie in the interval $0 \le f(E) \le 2$ and we give the function $f(E)$ as in equation (32).

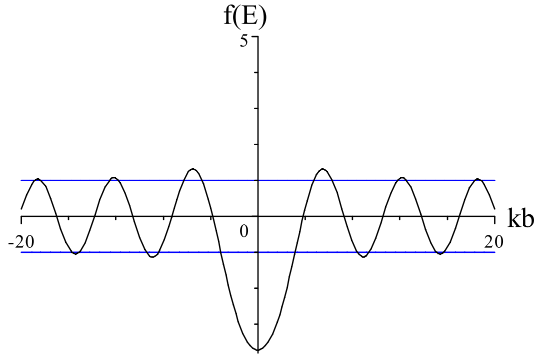

Fig. 7. Allowed energy eigenvalue spectrum at the limit $2\epsilon a \to 0$ and $V_0 \to -\infty$. Solid curve is drawn with $\Theta = 3\pi/2$.

(ii) The limit $2\epsilon a \to 0$, $V_0 = -\mu\omega^2 a^2/2 \to -\infty$: in this limit, the energy function $f(E)$ becomes

$$f(E) = \cos kb - \Theta \frac{\sin kb}{kb} \cdot \qquad (36)$$

From Figure 7, it can be seen that the energy spectrum consists of a series of separate regions, inside each of which the energy of the particle can vary continuously. These regions contains allowed and forbidden bands. The allowed bands increase as the energy increases. The results which we obtained in this limit are similar the band structure that has been derived with Kronig-Penney model.

5 Discussion

In this paper, we investigated the energy eigenvalues and eigenfunctions for one of the oldest problems of physics. Since the problem appears in optics, cosmology and solid state physics, to understand the quantum dynamics of this problem is important. For this purpose, firstly we have investigated quantum mechanically the brachistochrone problem in a linear gravity potential $-mgy$, and obtained the wave functions and energy spectrum of a particle with mass m.

Figure 1 shows the first eight energy levels of the system as a function of the parameter $\mu\omega a^2/\hbar$. These energy levels are not equally spaced for the small values of $\mu\omega a^2/\hbar$; the spaces between the energy eigenvalues increase with the quantum number n. Although the p_n values in equation (15) are not the exact analytic expression for the energy eigenvalues of the system, they reflect the behaviour of the problem. We can say that all p_n values are finite. On the other hand, the problem combines the infinite-well and harmonic oscillator potentials and reduces to the harmonic oscillator for $\omega \to \infty$ and to the infinite-well potential for $\omega \to 0$. The solutions reflect the properties of the energy spectrum that is proportional to n and n^2. The eigenfunctions have even- and odd-parity. In this work, we investigate the least time problem by the transformation $\varsigma = \sqrt{2\mu\omega/\hbar}\, a \cos\frac{\theta}{2}$, and obtained the solutions with this variable. We have also considered the classically allowed region as the interval $[-\sqrt{2\mu\omega/\hbar}\, a, \sqrt{2\mu\omega/\hbar}\, a]$, the potential is infinite at the outside of this interval.

Secondly, we have shown that the band structure arise from Floquet theory if the potential is periodic. For the limit $b \to 0$, the band structures of positive and negative energy states are similar to each other. For the positive energies and the limit $2\epsilon a \to 0$, $V_0 \to -\infty$, we have seen that our problem is equivalent to the periodic δ-potential problem or Kronig-Penney model.

This research is partially supported by Akdeniz University (20.01.0105.12).

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